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Շմավոնյան Հովհաննես Աշոտի

Օսցիլյատոր և Կուլոն համակարգերի սուպերինտեգրվող դեֆորմացիաները

Ա.04.02 - «Տեսական ֆիզիկա» մասնագիտությամբ
ֆիզիկամաթեմատիկական գիտությունների թեկնածուի
գիտական աստիճանի հայցման ատենախոսության

ՍԵՂՄԱԳԻՐ

ԵՐԵՎԱՆ-2019

A. I. ALIKHANYAN NATIONAL SCIENCE LABORATORY
(YEREVAN PHYSICS INSTITUTE)

Hovhannes Shmavonyan

Superintegrable deformations of oscillator and Coulomb systems

SYNOPSIS

of Dissertation in 01.04.02 - Theoretical Physics
presented for the degree of candidate in physical and mathematical
sciences

YEREVAN-2019

Ատենախոսության թեման հաստատվել է Ա. Ալիխանյանի անվան Ազգային Գիտական Լաբորատորիայի (ԵրՖԻ) գիտական խորհրդում:

Գիտական ղեկավար՝ Ֆիզմաթ. գիտ. դոկտոր Արմեն Ներսեսյան (ԱԱԳԼ)
Պաշտոնական ընդդիմախոսներ՝

ֆ.մ.գ.դ. Ռ. Պողոսյան (ԱԱԳԼ)
ֆ.մ.գ.դ. Լ. Մարդոյան (ԵՊՀ)

Առաջատար կազմակերպություն՝ Թբիլիսիի Պետական Համալսարանի Բարձր Էներգիաների Ֆիզիկայի Ինստիտուտ

Պաշտպանությունը կայանալու է 2019 թ մայիսի 24-ին ժամը 14-ին, ԱԱԳԼ-ում գործող ԲՈՀ-ի 024 «Ֆիզիկայի» մասնագիտական խորհրդում (Երևան - 0036, Ալիխանյան եղբայրների փ.2):
Ատենախոսությանը կարելի է ծանոթանալ ԱԱԳԼ գրադարանում:
Սեղմագիրը առաքված է 2019 թ. ապրիլի 13-ին:
Մասնագիտական խորհրդի գիտական քարտուղար՝

Ֆիզմաթ. գիտ. դոկտոր Հ. Մարուքյան

The subject of the dissertation is approved by the scientific council of the A. Alikhanyan National Science Laboratory (YerPhI)

Scientific advisor: Doctor of Sciences (Physics) Armen Nersessian (AANL)
Official opponents

Doctor of Sciences (Physics) R. Poghosyan (AANL)
Doctor of Sciences (Physics) L. Mardoyan (YSU)

Leading Organization:
High Energy Physics Institute of Tbilisi State University
Defense takes place on 24^{th} of May 2019 at 14:00 on meeting of the special council of the Higher Attestation Commission of RA 024 "Physics", acting in A.Alikhanyan National Science Laboratory (Yerevan-0036, st. Alikhanyan Brothers 2)
The dissertation can be found at the AANL library.
The synopsis is sent out on 13^{th} of April 2019.
Scientific secretary of the special council
Doctor of Sciences (Physics) H. Marukyan

Abstract

In this thesis first of all we present the holomorphic factorization formalism. This formalism allows to describe generalizations of Coulomb and oscillator models via introduction of complex variables. First of all we discuss this scheme on well known examples of TTW and PW systems. Then we do this for higher dimensional cases. We do the so called oscillator-Coulomb reduction procedure using the holomorphic factorization formalism. Moreover we discuss also curved spaces namely the spherical and pseudospherical generalizations. Finally we describe some examples of superintegrable models using this formalism.

After this we concentrate on the complex analogue of the Smorodinsky-Winternitz system interacting with an external magnetic field. Firstly we discuss the usual real N-dimensional Smorodinsky-Winternitz system. The main result we have obtained for the real case is the convenient form of the symmetry algebra. Then we introduce the complex analogue of this system, and write down the its hidden symmetries. We also obtain important result for this model, namely the symmetry algebra and quantum solutions. Eventually we compute the symmetry algebra for the generalized MICZ-Kepler system using the results we have obtained before for the $\mathbb{C}^2$-Smorodinsky-Winternitz system.

Moreover we introduce the complex projective analogue of the Rosochatius system in an external magnetic field. Here again we see that it is superintegrable, since it has hidden constants of motion. We write have found also its symmetry algebra, classical and quantum solutions. Namely we find solutions for the classical equations of motion, wavefunctions and the energy spectrum.

Finally we formulate the Supersymmetric generalizations of $\mathbb{C}^N$- Smorodinsky -Winternitz and $\mathbb{CP}^N$-Rosochatius models. For this purpose we have introduced $SU(2|1)$ supersymmetrization which allows to construct weak $\mathcal{N}=4$ superextensions of systems on Kähler manifolds interacting with an external magnetic field. We introduce $SU(2|1)$-Supersymmetric mechanics. Using this formalism one can find supersymmetric models on Kähler manifolds using the fact that all these systems can be viewed as $SU(2|1)$-Kähler oscillator with different Kähler potentials. Then we have shown Kähler potentials which give rise to $SU(2|1)$-Supersymmetric extensions of $\mathbb{C}^N$-Smorodinsky-Winternitz and $\mathbb{CP}^N$-Rosochatius systems.

Relevance and motivation

Integrable models are crucial for modern theoretical and mathematical physics. Due to the fact that different physical phenomena can have similar mathematical description, exactly solvable models can be used in many different areas. One can see that using these models huge amount of (both macroscopic and microscopic) physical phenomena can be described. Moreover integrable models can have applications even in other disciplines, due to the fact that system of integrable differential equations arise in other subjects e.g. mathematics, computer science, biology etc. The thesis is devoted to superintegrable extensions of oscillator and Coulomb models with an inverse square potential. Integrable models with inverse square potential are studied for few decades. Due to this fact they are well studied and there are many important results about these systems. Namely the Calogero-model has unique properties and due to that nowadays this is an important system in mathematical physics. On the other hand projective spaces have also interesting properties . Due to the fact that they are maximally symmetric spaces it is important to consider physical systems on these spaces. Unfortunately these two branches of mathematical physics are disconnected now. Complex analogs of Calogero model are not studied well and attempts to construct complexification of Calogero like models haven't succeeded yet.

Aim of the dissertation

Main goal of this thesis is to construct superintegrable generalizations of oscillator and Coulomb with an inverse square potential. Develop new formalism and study the properties of constructed models. We also focus on generalizations on Kähler manifolds ($\mathbb{C}^N$ and $\mathbb{CP}^N$) of oscillator with an inverse square potential.

- Develop a holomorphic factorization formalism, which is a convenient framework for constructing and working with oscillator and Coulomb like models namely with generalizations with an inverse square potential.
- Introduce complex generalization of Smorodinsky-Winternitz system. This is a generalization of oscillator with an inverse square potential. We also study the properties of this system. Also our is to find its constants of motion and solve it.
- Introduce a non-trivial superintegrable model on a complex-projective space, namely the analogue of Rosochatius model on complex projective space, find its conserved quantities, and explicit solutions
- Via presenting the technique of $SU(2|1)$-Supersymmetric mechanics we show that supersymmetric extensions of these systems can be constructed.

Novelty of the works

Formalism of holomorphic factorization is developed. Holomorphic factorization is done for TTW and PW systems. We also showed the relationship of constants of motion found via this formalism and the result known before. Several results for oscillator and Coulomb like models are rewritten using this formalism.Using this formalism isospectral models of Calogero model are found here.

Many crucial results are found here also for Smorodinsky-Winternitz model. As we know even for real case symmetry algebra of Smorodinsky Winternitz model was not written in a convenient form. Introduction of complex version in presence of an external magnetic field is also novelty. Moreover symmetry algebra and quantum solutions for the complex Smorodinsky model is also done here. Another important result is the derivation of the symmetry algebra of generalized MICZ-Kepler system.

Another important novelty is the introduction of superintegrable model with an inverse square potential on complex projective space ($\mathbb{CP}^N$-Rosochatius system). Conserved quantities of this model and the algebra they form is also an important result for us. Needless to say that classical and quantum solutions of this model is another important result.

Although formalism of Kähler superoscillator was introduced before, we gave explicit expressions of the Kähler potential, which give rise to the supersymmetric generalization of complex Smorodinsky and complex projective Rosochatius models.

Practical value

- Holomorphic factorization formalism can be used to investigate oscillator-like and Coulomb-like models, namely the generalization related to the Calogero model.

- In condensed matter physics models on complex projective spaces are strongly related with the quantum Hall effect. Particularly four-dimensional Hall effect can be related with the systems in $\mathbb{CP}^3$.

- In High energy physics their role cannot be overestimated. These systems can be viewed as simplified toy models for field theoretical complicated models in high energy research. Our particular example of Calogero model is an example of conformal mechanics. It is well known that conformal symmetry has a crucial role in modern high energy research. In this context supersymmetrization of these systems is also important. Moreover Calogero-like models are strongly related with AdS_2/CFT_1 correspondence. For this purpose supersymmetric extensions also have a crucial practical value.

- $\mathbb{C}^1$-Smorodinsky-Winternitz system is a popular model for the qualitative study of the so-called quantum ring , and the study of its behaviour in external magnetic field is quite a natural task. Respectively, $\mathbb{C}^N$-Smorodinsky-Winternitz could be viewed as an ensemble of N quantum rings interacting with external magnetic field. So investigation of its symmetry algebra is of the physical importance.

- $\mathbb{CP}^N$-Rosochatius system does not split into a set of N two-dimensional decoupled systems. Instead, it can be interpreted as describing *interacting* particles with a position-dependent mass in the two-dimensional quantum rings.

Content

CHAPTER 1

This chapter is an introduction and we introduce some general concepts are discussed. We give a brief discussion of Hamiltonian mechanics. We discuss well known examples of maximally superintegrable models, namely the oscillator and Coulomb systems. We discuss mechanical models interacting with an external magnetic field. and introduce action angle variables. Moreover we give a short review on Kähler manifolds and discuss maximally symmetric examples of it, namely complex Euclidean and complex projective spaces. Also a short description of supersymmetric mechanics is given.

CHAPTER 2

The goal of this chapter is to present "holomorphic factorization' to the superintegrable generalizations of oscillator and Coulomb systems on N-dimensional Euclidean space, sphere and two-sheet hyperboloid (pseudosphere). For this purpose we parameterize the phase spaces of that system by the complex variable $Z = p_r + i\sqrt{2\mathcal{I}}/r$ identifying the radial phase subspace with the Klein model of Lobachevsky plane, and by the complex variables $u_a = \sqrt{I_a}e^{i\Phi_a}$ unifying action-angle variables of the angular part of the systems.

Analyzing these deformations in terms of action-angle variables, it was found that they are superintagrable iff the spherical part has the form

$$\mathcal{I} = \frac{1}{2}\left(\sum_{a=1}^{N-1} k_a I_a + c_0\right)^2 \tag{1}$$

with c_0 be arbitrary constant and k_a be rational numbers . So different angular parts give different oscillator-like and Coulomb-like models. We formulate, in these terms, the constants of motion of the systems under consideration and calculate their algebra. Besides, we extend to these systems the known oscillator-Coulomb duality transformation.

The Trembley-Turbiner-Winternitz (TTW) system, invented a few years ago,. is a particular case of the Calogero-oscillator system. It is defined by the Hamiltonian of two-dimensional oscillator, with the angular part replaced by a Pöschl-Teller system on circle: Although the TTW and PW systems are particular cases of the Calogero-type models, they continue to attract enough interest due to their simplicity. In particular, a couple of years ago, it was suggested a specific representation for the constants of motion of the TTW and PW systems (including those on sphere and hyperboloid) , called a "holomorphic factorization". conserved quantities are presented and the symmetry algebra can also be computed.

So now we will briefly present some results we have found using this formalism. Holomorphic variables obey the following Poisson brackets:

$$\{Z, \bar{Z}\} = -\frac{\imath(Z-\bar{Z})^2}{2\sqrt{2\mathcal{I}}}, \qquad \{u_a, \bar{u}_b\} = -\imath\delta_{ab}, \tag{2}$$

$$\{Z, u_a\} = -u_a\Omega_a\frac{\imath(Z-\bar{Z})}{2\sqrt{2\mathcal{I}}}, \qquad \{Z, \bar{u}_a\} = \bar{u}_a\Omega_a\frac{\imath(Z-\bar{Z})}{2\sqrt{2\mathcal{I}}}, \tag{3}$$

where $\Omega_a = \Omega_a(I) = \frac{\partial\sqrt{2\mathcal{I}}}{\partial I_a}$.For conformal mechanics $\mathcal{H}_0 = \frac{p_r^2}{2} + \frac{\mathcal{I}}{r^2}$ we can write conserved quantities

$$\{Ze^{\imath\Lambda}, \mathcal{H}_0\} = 0 \qquad \text{iff} \qquad \{\Lambda, \sqrt{2\mathcal{I}}\} = -1. \tag{4}$$

where Λ is the angle-like variable, conjugate with $\sqrt{2\mathcal{I}}$.

Evidently, its real part is the ratio of Hamiltonian and its angular part and does not contain any new constant of motion. Nevertheless, such a complex representation seems to be useful not only from an aesthetical viewpoint, but also for the construction of supersymmetric extensions.

Note that we can write down the hidden symmetry generators for the conformal mechanics, modified by the oscillator and Coulomb potentials as well. The Hamiltonian of the N-dimensional oscillator and its hidden symmetry generators look as follows:

$$\mathcal{H}_{osc} = \mathcal{H}_0 + \frac{\omega^2 r^2}{2}, \quad \mathcal{M}_a^{osc} = \left(Z^2 - \frac{2\omega^2\mathcal{I}}{(\bar{Z}-Z)^2}\right)^{n_a} u_a^{2m_a}, \tag{5}$$

The Hamiltonian and hidden symmetry of the Coulomb problem are defined by

$$\mathcal{H}_{Coul} = \mathcal{H}_0 - \frac{\gamma}{\sqrt{r}}, \qquad \mathcal{M}_a^{Coul} = \left(Z - \frac{\imath\gamma}{2\sqrt{\mathcal{I}}}\right)^{n_a} u_a^{m_a}. \tag{6}$$

where

$$k_a = \frac{n_a}{m_a}, \qquad m_a, n_a \in \mathcal{N}. \tag{7}$$

So, we extended the method of "holomorphic factorization" initially developed for the two-dimensional oscillator and Coulomb system, to the superintegrable generalizations of Coulomb and oscillator systems in any dimension. For this purpose we parameterized the angular parts of these systems by action-angle variables. To our surprise, we were able to get, in these general terms, the symmetry algebra of these systems. Notice, that above formulae hold not only on the Euclidean spaces, but for the more general one, if we choose $\mathcal{I}$ be the system with a phase space different from T_*S^{N-1}.

We describe oscillator-Coulomb correspondence in terms complex variables introduced here. We will use "untilded" notation for the description of oscillator, and the "tilded" notation for the description of Coulomb system.

$$\tilde{Z} = \frac{\imath(\bar{Z}-Z)}{\lambda\sqrt{\mathcal{I}}}Z, \quad \tilde{\mathcal{I}} = \frac{\mathcal{I}}{4}, \quad \tilde{\Lambda} = 2\Lambda$$

Moreover we have considered spherical and pseudospherical deformations of these systems and found conserved quantities for above mentioned models on these curved spaces using the so called κ-deformation formalism. Holomorphic variable has the following form

$$Z = \sqrt{|\kappa|}\frac{p_\chi}{\sqrt{2}} + \frac{\imath\sqrt{\mathcal{I}}}{T_\kappa}, \tag{8}$$

Moreover deformed symmetry algebra is also found.

Examples of the spherical parts are also discussed, namely for

$$\mathcal{I} \equiv \frac{1}{2} j_N, \qquad j_a = p^2_{a-1} + \frac{j_{a-1}}{\sin^2 k_{a-1}\theta_{a-1}}, \qquad a = 1, \dots N-1 \tag{9}$$

and

$$\mathcal{I} = \frac{1}{2} F_{n-1}, \quad F_a = P^2_{\theta_a} + \frac{g^2_{a+1}}{\cos^2\theta_a} + \frac{F_{a-1}}{\sin^2\theta_a} \tag{10}$$

We have found u_a for these systems, so the holomorphic description can be easily used.

CHAPTER 3

The one-dimensional singular oscillator is a textbook example of a system which is exactly solvable both on classical and quantum levels.The sum of its N copies, i.e. N-dimensional singular isotropic oscillator is, obviously, exactly solvable as well. It is given by the Hamiltonian

$$H = \sum_{i=1}^{N} I_i, \qquad I_i = \frac{p_i^2}{2} + \frac{g_i^2}{2x_i^2} + \frac{\omega^2 x_i^2}{2}, \tag{11}$$

It is not obvious that in addition to Liouville Integrals I_i this system possesses supplementary series of constants of motion, and is respectively, *maximally superintegrable*, i.e. possesses $2N-1$ functionally independent constants of motion. All these constants of motion are of the second order on momenta. It seems that this was first noticed by Smorodinsky and Winternitz. For this reason this model is sometimes called

Smorodinsky-Winternitz system and we will use this name as well. Notice also that Smorodinsky-Winternitz system is a simplest case of the generalized Calogero model(with oscillator potential) associated with an arbitrary Coxeter root system. Thus, one hopes that observations done in this simple model could be somehow extended to the Calogero models. There is a well-known superintegrable generalization of the oscillator to sphere, which is known as Higgs oscillator.

About fifty years ago it was noticed that this system possesses additional set of constants of motion given by the expressions

$$I_{ij} = L_{ij}L_{ji} - \frac{g_i^2 x_j^2}{x_i^2} - \frac{g_j^2 x_i^2}{x_j^2}, \qquad \{I_{ij}, H\} = 0, \tag{12}$$

where L_{ij} are the generators of $SO(N)$ algebra.

The generators I_{ij} provides additional $N-1$ functionally independent constants of motions and so this system is maximally superintegrable. These generators define highly nonlinear symmetry algebra,

$$\{I_i, I_{jk}\} = \delta_{ij}S_{ik} - \delta_{ik}S_{ij}, \quad \{I_{ij}, I_{kl}\} = \delta_{jk}T_{ijl} + \delta_{ik}T_{jkl} - \delta_{jl}T_{ikl} - \delta_{il}T_{ijk} \tag{13}$$

where

$$S^2_{ij} = -16(I_iI_jI_{ij} + I_i^2 g_j^2 - I_j^2 g_i^2 + \frac{\omega^2}{4} I^2_{ij} - g_i^2 g_j^2 \omega^2) \tag{14}$$

$$T^2_{ijk} = -16(I_{ij}I_{jk}I_{ik} + g_k^2 I^2_{ij} + g_j^2 I^2_{ik} + g_i^2 I^2_{jk} - 4g_i^2 g_j^2 g_k^2). \tag{15}$$

The generators S^2_{ij} and T^2_{ijk} are of the sixth-order in momenta and antisymmetric over i, j, k indices. The above symmetry algebra could be written in a compact form if we redefine the generators. Quantum-mechanically the maximal superintegrability is reflected in the dependence of its energy spectrum on the single,"principal" quantum number only.

Moreover we consider simple generalization of the Smorodinsky-Winternitz system *interacting with constant magnetic field.* It is defined on the N-dimensional complex Euclidian space parameterized by the coordinates z^a by the Hamiltonian

$$\mathcal{H} = \sum_{a=1}^{N} \left(\pi_a \bar\pi_a + \frac{g_a^2}{z^a \bar z^a} + \omega^2 z^a \bar z^a \right), \qquad \{\pi_a, z^b\} = \delta_{ab}, \quad \{\pi_a, \bar\pi_b\} = \imath B \delta_{ab} \tag{16}$$

The (complex) momenta π_a have nonzero Poisson brackets due to the presence of magnetic field with magnitude B. We will refer this model as

$\mathbb{C}^N$-Smorodinsky-Winternitz system.It can be interpreted as a sum of N two-dimensional singular oscillators interacting with constant magnetic field perpendicular to the plane. For sure, in the absence of magnetic field this model could be easily reduced to the conventional Smorodinsky-Winternitz model, but the presence of magnetic field could have nontrivial impact which is studied in this chapter. So, *our main goal is to investigate the whole symmetry algebra of this system.* Set of hidden constants of motion defined in complete analogy with those of conventional Smorodinsky-Winternitz system:

$$I_{ab} = L_{a\bar{b}}L_{b\bar{a}} + \Big(\frac{g_a^2 z^b \bar{z}^b}{z^a \bar{z}^a} + \frac{g_b^2 z^a \bar{z}^a}{z^b \bar{z}^b}\Big), \quad \{I_{ab}, \mathcal{H}\} = 0, \qquad a \neq b \qquad (17)$$

with $L_{a\bar{b}}$ being generators of $SU(N)$ algebra. Also symmetry algebra is computed in this chapter, which has more complicated form, but after redefinition it will have the similar form to the symmetry algebra of the real Smorodinsky-Winternitz model with redefined symmetry generators.

Let us briefly discuss the number of conserved quantities. We have N real functionally independent constants of motion (I_a). Moreover let us mention that I_{ab} is also real, and although it has $N(N-1)/2$ components, the number of functionally independent constants of motion is $N-1$.

In addition to this, the complex system has N real conserved quantities $(L_{a\bar{a}})$. So the total number of constants of motion is $3N-1$ and it is superintegrable (but not maximally superintegrable). Especially if $N=1$ the system is integrable. For $N=2$ the system is superintegrable, but it has only one additional constant of motion (minimally superintegrable).

Quantization will be done using the fact that $\mathbb{C}^N$-Smorodinsky-Winternitz system is a sum of two dimensional singular oscillators. This allows to write the wave function as a product of N wave functions and total energy of the system as a sum of the energies of its subsystems. So the initial problem reduces to two-dimensional one. After solving the Schrödinger equation one can find the energy spectrum and wavefunction of this system. The wavefunction contains a hypergeometric function. As it was mentioned in contrast to the real case it depends on $N+1$ quantum numbers, namely n and m_a .

$$E_{tot} = \sum_{a=1}^{N} E_{n_a, m_a} = \hbar\sqrt{\omega^2 + \frac{B^2}{4}}\Big(2n+N+\sum_{a=1}^{N}\sqrt{m_a^2 + \frac{4g_a^2}{\hbar^2}}\Big) + \frac{B\hbar}{2}\sum_{a=1}^{N} m_a, \qquad (18)$$

Since $\mathbb{C}^N$-Smorodinsky-Winternitz system has manifest $U(1)$ invariance, we could apply its respective reduction procedure related with first Hopf map $S^3/S^1 = S^2$, which is known as Kustaanheimo-Stiefel transformation, for the particular case of $N=2$. Such a reduction was performed decade ago and was found to be resulted in the so-called "generalized MICZ-Kepler problem" suggested by Mardoyan a bit earlier. However the initial system was considered, it was not specified by the presence of constant magnetic field, furthermore, the symmetry algebra of the reduced system was not obtained there. Hence, it is at least deductive to perform Kustaanheimo-Stiefel transformation to the $\mathbb{C}^2$-Smorodinsky-Winternitz system with constant magnetic field in order to find its impact (appearing in the initial system) in the resulting one. Furthermore, it is natural way to find the constants of motion of the "generalized MICZ-Kepler system" and construct their algebra.

For this purpose we have to choose six independent functions of initial phase space variables which commute with that generators,

$$q_k = z\sigma_k\bar{z}, \quad p_k = \frac{z\sigma_k\pi + \bar{\pi}\sigma_k\bar{z}}{2z\bar{z}}, \quad k = 1,2,3 \qquad (19)$$

where σ_k are standard 2×2 Pauli matrices. Matrix indices are dropped here. This transformation is called Kustaanheimo-Stiefel transformation.To get the Coulomb-like system we fix the energy surface or reduced Hamiltonian, $H_{SW} - E_{SW} = 0$ and divide it on $2|q|$. Then we get the expression, which defines the Hamiltonian of "generalized MICZ-Kepler problem". Hence, we transformed the energy surface of the reduced $\mathbb{C}^2$-Smorodinsky-Winternitz Hamiltonian to those of (three-dimensional) "Generalized MICZ-Kepler system". Additionally it has an inverse square potential and this system has an interaction with a Dirac monopole magnetic field which affects the symplectic structure.

$$\mathcal{H}_{gMICZ} = \frac{p^2}{2} + \frac{s^2}{2|q|^2} + \frac{g_1^2}{|q|(|q|+q_3)} + \frac{g_2^2}{|q|(|q|-q_3)} - \frac{\gamma}{|q|} \qquad (20)$$

Surprisingly, the reduced system contains interaction with Dirac monopole field only, i.e. the constant magnetic field in the original system does not contribute in the reduced one. All dependence on B is hidden in s and γ, which are fixed, so the reduced system does not depend on B explicitly.

One can also perform the reduction for conserved quantities and we can find that

$$\mathcal{I} = \frac{I_1 - I_2}{2} + \frac{B}{4}(L_{22} - L_{11}), \quad \mathcal{L} = \frac{1}{2}(L_{22} - L_{11}), \quad \mathcal{J} = I_{12}, \tag{21}$$

where non-calligraphic letters are the symmetry generators of the initial $\mathbb{C}^2$-Smorodinsky-Winternitz model. Using this relationship we also found the symmetry algebra of the generalized-MICZ-Kepler system. The only non-zero commutator is the following one

$$\{\mathcal{I}, \mathcal{J}\} = S \tag{22}$$

where

$$\begin{aligned} S^2 = 2\mathcal{H}_{gMICZ}\Big[4\Big(\mathcal{J} + \frac{1}{2}\big(\mathcal{L}^2 - s^2\big)\Big)^2 - \Big(4g_2^2 + (\mathcal{L}+s)^2\Big)\Big(4g_1^2 + (\mathcal{L}-s)^2\Big)\Big] \\ - \Big(4g_2^2 + (\mathcal{L}+s)^2\Big)\Big(\mathcal{I}+\gamma\Big)^2 - \Big(4g_1^2 + (\mathcal{L}-s)^2\Big)\Big(\mathcal{I}-\gamma\Big)^2 \\ - 4\Big(\mathcal{J} + \frac{1}{2}(\mathcal{L}^2 - s^2)\Big)\Big(\mathcal{I}-\gamma\Big)\Big(\mathcal{I}+\gamma\Big) \end{aligned} \tag{23}$$

CHAPTER 4

The maximally superintegrable spherical counterpart of the Smorodinsky-Winternitz system is defined by the Hamiltonian suggested by Rosochatius in 1877. It is a particular case of the integrable systems obtained by restricting the free particle and oscillator systems to a sphere. It was studied by many authors from different viewpoints, including its re-invention as a

superintegrable spherical generalization of Smorodinsky-Winternitz system. Rosochatius model, as well as its hybrid with the Neumann model, attract a stable interest for years due to their relevance to a wide circle of physical and mathematical problems. Recently, the Rosochatius-Neumann system was encountered, while studying strings, extreme black hole geodesics and Klein-Gordon equation in curved backgrounds.
In this chapter we propose a superintegrable generalization of Rosochatius (and Smorodinsky-Winternitz) system on the complex projective space $\mathbb{CP}^N$. It is defined by the Hamiltonian

$$\mathcal{H}_{Ros} = (1+z\bar{z})\frac{(\pi\bar{\pi}) + (z\pi)(\bar{z}\bar{\pi})}{r_0^2} + r_0^2(1+z\bar{z})(\omega_0^2 + \sum_{a=1}^{N}\frac{\omega_a^2}{z^a\bar{z}^a}) - r_0^2\sum_{i=0}^{N}\omega_i^2, \tag{24}$$

and by the Poisson brackets providing the interaction with a constant magnetic field of the magnitude B

$$\{\pi_a, z^b\} = \delta_a^b, \quad \{\bar{\pi}_a, \bar{z}^b\} = \delta_{\bar{a}}^{\bar{b}}, \quad \{\pi_a, \bar{\pi}_b\} = \imath B r_0^2\left(\frac{\delta_{a\bar{b}}}{1+z\bar{z}} - \frac{\bar{z}^a z^b}{(1+z\bar{z})^2}\right). \tag{25}$$

We will call it $\mathbb{CP}^N$-*Rosochatius system*. Rescaling the coordinates and momenta as $r_0 z^a \to z^a, \pi_a/r_0 \to \pi_a$ and taking the limit $r_0 \to \infty, \omega_a \to 0$ with $r_0^2\omega_a = g_a$ kept finite, we arrive at the $\mathbb{C}^N$-Smorodinsky-Winternitz system discussed in the previous chapter. The model has N manifest (kinematical) $U(1)$ symmetries with the generators and hidden symmetries with the second-order generators $I_{ij} = (I_{0a}, I_{ab})$ defined as

$$I_{0a} = J_{0a}\bar{J}_{0\bar{a}} + \omega_0^2 z^a\bar{z}^a + \frac{\omega_a^2}{\bar{z}^a z^a}, \qquad I_{ab} = J_{a\bar{b}}J_{b\bar{a}} + \omega_a^2\frac{z^b\bar{z}^b}{z^a\bar{z}^a} + \omega_b^2\frac{z^a\bar{z}^a}{z^b\bar{z}^b}. \tag{26}$$

For sure, the symmetry algebra written above can be found by a direct calculation of the Poisson brackets between the symmetry generators. However, there is a more elegant and simple way to construct it. Namely, one has to consider the symmetry algebra of $\mathbb{C}^{N+1}$-Smorodinsky-Winternitz system (Part III) with *vanishing* magnetic field, and to reduce it, by action of the generators $\imath(p_i u^i - \bar{p}_i\bar{u}^i)$, $u^i\bar{u}^i$, to the symmetry algebra of $\mathbb{CP}^N$-Rosochatius system.

As was already noticed, the $\mathbb{S}^N$-Rosochatius system is maximally superintegrable, *i.e.* it has $2N-1$ functionally independent constants of motion. On the other hand the $\mathbb{CP}^N$-Rosochatius system has $2N - 1 + N =$

$3N-1$ functionally independent integrals. Hence, it lacks N integrals needed for the maximal superintegrability. This situation is similar to the Smorodinsky-Winternitz system, which is not surprising since it is the flat limit of the Rosochatius model. We present the Hamiltonian in these coordinates and the angular part has the form of the Pöschl-Teller potential. Now we have the spherical coordinates and in this coordinate system separations of variables can be done. After this we write classical solutions and wavefunction and the energy spectrum. It is worth to mention that the wavefunction is written in terms of the hypergeometric function, like for the Smorodinsky-Winternitz system. As was expected the energy spectrum depends on $N+1$ quantum numbers.

CHAPTER 5

Now we want to construct $\mathcal{N}=4$ supersymmetric generalizations of $\mathbb{C}^N$-Smorodinsky-Winternitz and $\mathbb{CP}^N$-Rosochatius systems. As was mentioned $U(1)$ invariant structure of these models allowed us to introduce an external constant magnetic field without violating the symmetries of the initial system. So for supersymmetric extensions of these models we want to keep this interaction.

We define the supercharges and the symplectic structure of the $SU(2|1)$-Superoscillator in the following form

$$Q^\alpha = e^{i\nu/2}\cos\lambda\Theta^\alpha + e^{-i\nu/2}\sin\lambda\epsilon^{\alpha\gamma}\overline{\Theta}_\gamma \qquad (27)$$

$$\{\pi_a, z^b\} = \delta^b_a, \quad \{\pi_a, \eta^b_\alpha\} = -\Gamma^b_{ac}\eta^{c\alpha},$$

$$\{\pi_a, \bar\pi_b\} = i(Bg_{a\bar b} + iR_{a\bar b c\bar d}\eta^{c\alpha}\bar\eta^d_\alpha), \quad \{\eta^{a\alpha}, \bar\eta^b_\beta\} = g^{a\bar b}\delta^\alpha_\beta. \qquad (28)$$

where

$$\Theta^\alpha = \pi_a\eta^{a\alpha} + \imath\bar\omega\bar\partial_a K\epsilon^{\alpha\beta}\bar\eta^a_\beta \qquad (29)$$

$$\cos 2\lambda = \frac{B}{\sqrt{4|\omega|^2+B^2}}, \quad \sin 2\lambda = -\frac{2|\omega|}{\sqrt{4|\omega|^2+B^2}}, \qquad \omega = |\omega|e^{\imath\nu} \qquad (30)$$

Now we can present the supersymmetic algebra $SU(2|1)$

$$\{Q^\alpha, Q^\beta\} = 0, \qquad \{\overline{Q}_\alpha, \overline{Q}_\beta\} = 0, \qquad (31)$$

$$\{Q^\alpha, \overline{Q}_\beta\} = \delta^\alpha_\beta\mathcal{H}_{osc} + \sqrt{4|\omega|^2+B^2}\,\mathcal{R}^\alpha_\beta \qquad \{Q^\alpha, \mathcal{R}^\beta_\gamma\} = -\imath\delta^\alpha_\gamma Q^\beta + \frac{\imath}{2}\delta^\beta_\gamma Q^\alpha \qquad (32)$$

$$\{Q^\alpha, \mathcal{H}_{osc}\} = \imath\sqrt{|\omega|^2+\frac{B^2}{4}}\,Q^\alpha \qquad \{\mathcal{R}^\alpha_\beta, \mathcal{H}_{osc}\} = 0 \qquad (33)$$

$$\{\mathcal{R}^\alpha_\beta, \mathcal{R}^\gamma_\delta\} = \imath\delta^\gamma_\beta\mathcal{R}^\alpha_\delta - \imath\delta^\alpha_\delta\mathcal{R}^\gamma_\beta. \qquad (34)$$

where R-charges and the Hamiltonian are as follows

$$\mathcal{R}^\alpha_\beta = \imath g_{a\bar b}\eta^{a\alpha}\bar\eta^b_\beta - \frac{\imath}{2}\delta^\alpha_\beta g_{a\bar b}\eta^{a\gamma}\bar\eta^b_\gamma \qquad (35)$$

$$\mathcal{H}_{osc} = g^{a\bar b}(\pi_a\bar\pi_b + |\omega|^2\partial_a K\bar\partial_b K) - \frac{1}{2}R_{a\bar b c\bar d}\eta^{a\alpha}\bar\eta^b_\alpha\eta^{c\beta}\bar\eta^d_\beta$$
$$+ \frac{\imath}{2}\omega K_{a;b}\eta^{a\alpha}\eta^b_\alpha + \frac{\imath}{2}\bar\omega K_{\bar a;\bar b}\bar\eta^a_\alpha\bar\eta^{b\alpha} + \frac{B}{2}\imath g_{a\bar b}\eta^{a\alpha}\bar\eta^b_\alpha. \qquad (36)$$

Let us remind that Kähler potential is defined up to (anti-)holomorphic function, so that the above superesymmetrization involves, not a single Hamiltonian, but a family of Hamiltonians parameterized by arbitrary holomorphic function. Now we present Kähler superoscillator for the $\mathbb{C}^N$-Smorodinsky-Winternitz system. We define it by the Kähler potential

$$K = z\bar z + \frac{g_a}{\omega}\log z^a + \frac{\bar g_a}{\bar\omega}\log\bar z^a. \qquad (37)$$

In that case the Hamiltonian decouples to the sum of N weak supersymmetric $\mathbb{C}^1$-Smorodinsky-Winternitz systems. On the other hand Kähler potential

$$K = \log(1-z\bar z) - \frac{1}{|\omega|}\sum_{a=1}^{N}(\omega_a\log z^a + \bar\omega_a\log\bar z^a), \quad \omega = \omega_0 + \sum_{a=1}^{N}\omega_a. \qquad (38)$$

generates supersymmetric extension for the $\mathbb{CP}^N$.

In **Conclusion** the main results are discussed.

Complete **Bibliography** is presented in the thesis.

Publication list

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Ամփոփում

- Ներկայացված է հոլոմորֆ ֆակտորիզացիայի ֆորմալիզմը: Այս ֆորմալիզմը թույլ է տալիս նկարագրել Կուլոն և օսցիլյատոր համակարգերը կոմպլեքս փոփոխականների ներմուծման միջոցով:
- Սկզբում մենք քննարկել ենք այս սխեման TTW և PW համակարգերի համար, այնուհետև ընդհանրացրել բարձր չափողականություն ունեցող համակարգերի համար:
- Կատարվել է օսցիլյատոր-Կուլոն ռեդուկցիան օգտագործելով հոլոմորֆ ֆակտորիզացիայի ֆորմալիզմը:
- Քննարկվել են այս համակարգերի ընդհանրացումները կորացած տարածությունների համար, մասնավորապես սֆերիկ և փսևդոսֆերիկ դեպքերը:
- Դիտարկվել են որոշ սուպերինտեգրվող համակարգերի օրինակներ օգտագործելով այս ֆորմալիզմը:
- Քննարկվել է բազմաչափ իրական Սմորոդինսկի-Վինտերնից համակարգը և ստացվել է սիմետրիայի հանրահաշիվը այս համակարգի համար:
- Ներմուծվել է Սմորոդնիկսի-Վինտերնից համակարգի կոմպլեքս համարժեք համակարգ , ներկայացվել պահպանվող մեծությունները, ինչպես նաև նրանց կազմած սիմետրիայի հանրահաշիվը:
- Քննարկվել է այս համակարգի քվանտացումը, ստացվել են ալիքային ֆունկցիան և էներգիայի սպեկտրը:
- Ստացված արդյունքների օգնությամբ հաշվվել է ընդհանրացված ՄԻԿՑ-Կեպլեր համակարգի սիմտրիայի հանրահաշիվը:
- Ներմուծվել է Ռոսոխատիուս համակարգը կոմպլեքս պրոյեկտիվ տարածությունների վրա, ներկայացվել են շարժման ինտեգրալները, սիմետրիայի հանրահաշիվը:
- Լուծելով Համոլտոն-Յակոբիի հավասարումները ստացվել են դասական լուծումները:

- Քննարկվել է կոմպլեքս պրոյեկտիվ Ռոսոխատիուս համակարգի քվանտացումը, գրվել է Շրոդինգերի հավասարումը և ստացվելեն ալիքային ֆունկցիաները ու էներգիայի սպեկտրը:
- Նկարագրված է N=4 թույլ Սուպերսիմետրիկ մեխանիկան Կելերյան տարածությունների վրա:
- Քննարկվել է SU(2|1) Լանդաուի խնդիրը այնուհետև Սուպերսիմետրիկ SU(2|1) Կելերյան օսցիլյատորը:
- Այս մոտեցման միջոցով կառուցվել են կոմպլեքս Սմորոդինսկի-Վինտերնից և կոմպլեքս պրոյոկտիվ Ռոսոխատիուս համակարգերի սուպերսիմետրիկ ընդհանրացումներ:

Резюме

- Представлен формализм голоморфной факторизации. Этот формализм позволяет нам описывать системы Кулона и осциллятора посредством введения комплексных переменных.
- Сначала мы обсуждали эту схему для систем TTW и PW, а затем обобщали для многомерных систем.
- Редукция осциллятора-Кулона была выполнена с использованием формализма голоморфной факторизации.
- Обсуждались обобщения этих систем для искривленных пространств, в частности, сферические и псевдосфорические случаи .
- Рассматривались некоторые примеры суперинтегрируемых систем с использованием формализма голоморфной факторизации.
- Обсуждалась вещественная многомерная система Смородинского-Винтерсница и была получена алгебра симметрии для этой системы.
- Был введенц комплексный аналог системы Смородинского-Винтерница, введены сохраняющийся виличины и образующийся алгебра симметрии.
- Обсуждаено квантование этой системы, получены волновая функция и энергетический спектр.
- Была вычислена алгебра симетрии обобщенной системы МИКЦ-Кеплер.
- Система Росохатиус была введена в комплексном проективном пространстве, били представлены интегралы движения, алгебра симметрии.
- Решая уравнения Гамолтона-Якоби были получены классические решения.
- Обсуждалось квантование системы. получены решения уравнения Шредингера. волновые функции и энергетический спектр.
- Описана слабая N = 4 суперсимметричная механика в Келлеровых пространствах.

- Сначала обсуждалась SU (2|1) задача Ландау, а затем суперсимметричный SU (2|1) осциллятор Кеплера.
- Благодаря такому подходу были построены суперсимметричные обобщения комплексаной системы Смородинского-Винтерница и комплексной проективной системы Россохатиуса.